DIFERENCIJALNI OPERATOR TIPA STURM-LIOUVILLE NA SEGMENTU [0, π] SA PROMJENLJIVIM KAŠNJENJEM

DIFFERENTIAL OPERATOR STURM- LIOUVILLE TYPE ON THE SEGMENT [0, π] WITH VARIABLE DELAY

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Stručni rad

REZIME
U ovom radu izučavana je diferencijalna jednačina drugog reda na segmentu $[0,\pi]$ sa promjenljivim kašnjenjem. Konstruisana je karakteristična funkcija operatora generisanog ovom jednačinom i rubnim uslovima na krajevima razmaka, izučavana određena poopštenja tih operatora, koja se prirodno nameću.

Ključne riječi:
karakteristična funkcija,
spektralna analiza,
asimptotika funkcije,
svojstvene vrijednosti

Professional paper

SUMMARY
This paper studies the differential equation of the second order in the segment [0, π] with varying delays. Construct a characteristic function of the operator generated by this equation and the boundary conditions at the ends of space, studying certain generalization of these operators, which naturally imposed.

Keywords:
characteristic functions,
spectrum analysis,
asymptotic functions,
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1. UVOD

Spektralna analiza predstavlja jednu modernu matematičku teoriju koja se pokazala kao izuzetno efikasna pri rješavanju jedne jako široke klase problema iz raznih naučnih disciplina kao što su matematika, mehanika, fizika, elektronika, geofizika, meteorologija i druge prirodne i tehničke nauke. Obrnuti spektralni problemi danas predstavljaju jedan od najpopularnijih dijelova spektralne analize čemu u prilog govori veliki broj radova posvećenih upravo ovoj problematici [8] i [9]. Najveći rezultati u spektralnoj teoriji u cjelini , a posebno za inverzne spektralne probleme postignut je za Sturm-Liouville-ov operator

$$L[y] = -y''(x) + q(x)y$$

koji se takođe naziva i jednodimenzionalni Schrödinger operator.

Ovaj matematički aparat našao je primjenu u matematici, mehanici, elektronici i drugim naukama [1] i[4].

1. INTRODUCTION

Spectral analysis is a modern mathematical theory that has proved to be extremely effective in addressing a much broader class of problems from different disciplines such as mathematics, mechanics, physics, electronics, geophysics, meteorology and other natural and technical sciences. Reverse spectral problems today represent one of the most popular parts of the spectral analysis which is supported by a large number of papers devoted to just this issue[8] i [9] . The greatest results in the spectral theory in general and specifically to inverse spectral problems has been made from integral Sturm-Liouvill operator

$$L[y] = -y''(x) + q(x)y$$

which is also called one-dimensional Schrödinger operator.

This mathematical model is found in the application of mathematics, mechanics, electronics and other sciences [1] and [4].

Ova klasa diferencijalnih operatora opisuje fizičke pojave koje ne zavise samo od trenutka kad se dešavaju već na njih značajno utiče i ono što se desilo i prije tog trenutka. Jedan od takvih problema je i tema ovog rada.
Spektralne karakteristike operatora $L[y]$ podrazumjevaju izučavanje asimptotike njegovih svojstvenih vrijednosti, asimptotike njegovih svojstvenih funkcija, razlaganje u red po svojstvenim funkcijama, rješavanje obratne zadaće i određivanje regularizovanih tragova. Posebnu klasu problema predstavljaju obrnuti problemi za operatore tipa Sturm-Liouville kod kojih se potencijal $q(x)$ i argumentna funkcija y ne javljaju sa istim argumentom. Diferencijalne jednačine koje opisuju ovakve operatore nazivaju se diferencijalne jednačine sa pomjerenim argumentom.

***Definicija1*:** Diferencijalna jednačina sa pomjerenim argumentom je ona diferencijabilna jednačina u kojoj se nepoznata funkcija i njeni izvodi pojavljuju za različite vrijednosti argumenta.
Pored niza graničnih problema koji su izučavani i publikovani dobijeni rezultati istraživanja, u ovom radu analiziram sljedeći granični problem dat u obliku diferencijalnog operatora

This class of differential operators describing the physical phenomena that do not depend only from the moment they are happening already on them is significantly affected by what happened before that moment. One such problem is the subject of this paper.
The spectral characteristics of the operator $L[y]$ implise asymptotic study of its intrinsic value, its asymptotic eigenfunctions, decomposition in order eigenfunctions, solving tasks and reverse the determination regular traces. A special class of inverse problems are problems for operators of Sturm-Liouville type where the potential $q(x)$ and the argument of the function y does not occur with the same argument . Differential equations that describe these operators are called differential equations with shifted argument.

Definition1.: Differential equations whit shifted argument it is a differential equation in which the unknown function and its running appear for different values of the argument.
Besides a number of boundary value problems which are studied and published the results od research, studied the following boundary problem

$$-y''(x) + q(x)y(x-\tau) + \sum_{i=1}^{k} p_i(x)y(\beta_i) = \lambda y(x) = z^2 y(x)$$

$$y(0) = y(\pi) = 0$$

$$y(x-\tau) = 0; \quad x \leq \tau;$$

Dakle, početni skup je segment $[-\tau, 0]$, koji se za $\tau > 0$ ne svodi na tačku. Ovaj uslov opisuje pojave koje se javljaju u raketnoj tehnici i balistici, neposredno prije ispaljivanja samog projektila i faktore koji utiču na brzinu i putanju kao i pojave u geofizici kod naprezanja i pomjeranja tektonskih ploča.

So, the initial set of the segment $[-\tau, 0]$, that for $\tau > 0$ can not be reduced to the point. This condition describes the phenomena that occur in missile technology and ballistics, just before firing the missiles, and the factors that affect the speed and trajectory as well as the phenomena in geophysics at the stress and displacement of tectonic plates.

2. POOPŠTENJE OPERATORA

U ovom radu posmatramo diferencijalni operator :

2. GENERALIZATION OPERATORS

In this paper, we consider the operator

$$-y''(x) + q(x)y\big(\alpha(x)\big) + \sum_{i=1}^{k} p_i(x)y(\beta_i) = \lambda y(x) = z^2 y(x) \tag{1}$$

$$y(0) = 0 \tag{2}$$

$$y(\pi) = 0 \tag{3}$$

pri čemu je operator za $p_i(x) \equiv 0 \;\; x \in [0,\pi]$ razmatran u [7] i konstruisana je njegova karakteristična funkcija. Početni skup svodi se na tačku $x = 0$. Problem se svodi na rješavanje integralne jednačine tipa Voltera, čije rješavanje dajemo u obliku sljedeće teoreme.

TEOREMA 1.: Ako su funkcije $q(x)$, $p_i(x)$, $(\; i = 1,2,\dots,n)$ i $\alpha(x) \; > \; 0, \; < \alpha'(x) < x, \; 0 < \alpha(x) < x$, za svako $x\epsilon(0,\pi]$, $\alpha(0) = 0$, $x\epsilon[0,\pi]$ beskonačno diferencijabilne funkcije i neka su $\beta_i(x)$, $x \in (0,\pi)$ $i = 1,2,\dots,k$ zadati ali proizvoljni brojevi. Tada je karakteristična funkcija $F_{\beta_1}(z)$ operatora zadatog od (1-3) meromorfna funkcija, čiji su polovi nule funkcije $z - \psi(\beta_i, z)$ i nalaze se u nekom krugu dovoljno velikog poluprečnika.

Dokaz : Neka je

whereby the operator of $p_i(x) \equiv 0 \;\; x \in [0,\pi]$ discussed in [7] and is construced its characteristic function. The intial set is reduced to the point $x = 0$. The problem boils down to solving the following equation ,the resolution of which are presented in the form of the following theorem.

Theorem: If the function $q(x)$, $p_i(x)$, $(\; i = 1,2,\dots,n)$ i $\alpha(x) > 0$ i $0 < \alpha'(x) < x$, $0 < \alpha(x) < x$. For each $x\epsilon(0,\pi], \alpha(0) = 0, x\epsilon[0,\pi]$ infinitely differentiable functions and some are $\beta_i(x)$ $(0,\pi)$ $i = 1,2,\dots,k$ set but arbitary numbers. Then the characteristic function $F_{\beta_1(}(z)$ the operator of the present (1-3) meromorphic function whose poles zero function.

Proof: Let

$$y(x,z) = 2icsinzx + \frac{1}{z}\int_0^x q(t)sinz(x-t)y(\alpha(t))dt + \sum_{i=1}^{k} y\beta_i \int_0^x p_i(t)sinz(x-t)dt$$

gdje je C-konstanta, $i = \sqrt{-1}$ i neka je

where C is a constant $i = \sqrt{-1}$ and let

$$h_i(x,z) = sinzx + \int_0^x p_i(t)sinz(x-t)dt \;\; ; \;\; i = 1,2,\dots,k$$

Bez ograničenja opštosti možemo staviti 2iC = 1.
Tako dolazimo do integralne jednačine.

without limiting the generality we can put $2iC = 1$.
So we come to the integral equation.

$$y(x,z) = sinzx + \sum_{i=1}^{k} \frac{1}{z} h_i(x,z) y(\beta_i, z) + \frac{1}{z}\int_0^x q(t)sinz(x-t)y(\alpha(t))dt \qquad (4)$$

Stavimo:

We'll put

$$y_0(x,z) = sinzx + \sum_{i=1}^{k} \frac{h_i(x,z)}{z} y(\beta_i, z)$$

Mi ćemo, smatrati veličine $y(\beta_i, z)$ $i = 1, \dots, k$, poznatim veličinama i jednačinu (4) možemo rješavati metodom uzastopnih aproksimacija.

Za $k = 1$ slijedi :

We will be considered as the size of $y(\beta_i, z)$ $i = 1, \dots, k$ known sizes and equation (4) can be solved by successive approximations.

For $k = 1$ follows:

$$y_1(x,z) = \left[sinzx + \frac{1}{z}\int_0^x q(t)\sin z(x-t) y(\alpha(t))dt \right] + y(\beta_1,z)\left[\frac{h_1(x,z)}{z} + \frac{1}{z}\int_0^x q(t)sinz(x-t)h_1(\alpha(t),z)dt\right]$$

$$y_2(x,z) = [sinzx + \frac{1}{z}\int_0^x q(t)\sin z(x-t)\sin z(\alpha(t))dt + \frac{1}{z}\int_0^x q(t)\sin z(x-t)\int_0^{\alpha(t)} q(t_1)\, sinz(\alpha(t)-t_1) sinz\alpha(t_1)dt_1dt \Bigg] + y(\beta_1,z)\left[\frac{h_1(x,z)}{z} + \frac{1}{z^2}\int_0^x q(t)sinz(x-t)h_1(\alpha(t),z)dt + \frac{1}{z^3}\int_0^x q(t)\sin z(x-t)\int_0^{\alpha(t)} q(t_1)\, sinz(\alpha(t)-t_1)h_1(\alpha(t_1),z)dt_1dt\right].$$

Uveli smo sljedeće oznake zbog jednostavnijeg zapisivanja:

Introduce the following tags:

$$\varphi(x,z) = sinzx + \sum_{i=1}^{\infty}\frac{1}{z^i}\left\{\int_0^x q(t)\sin z(x-t)\int_0^{\alpha(t)} q(t_1)sinz(\alpha(t)-t_1\int_0^{\alpha(t_1)} \dots \dots \int_0^{\alpha(t_i)} q(t_{i+1})\, sinz(\alpha(t_i)-t_{i+1})dt_{i+1}\dots dt_1dt\right\}$$

$$\psi(x,z) = h_1(x,z) + \sum_{i=1}^{\infty}\frac{1}{z^i}\left\{\int_0^x q(t)\sin z(x-t)\int_0^{\alpha(t)} q(t_1)sinz(\alpha(t)-t_1\int_0^{\alpha(t_1)} \dots \dots \int_0^{\alpha(t_i)} q(t_{i+1})\, sinz(\alpha(t_i)-t_{i+1})h_1(\alpha(t_{i+1},z)dt_{i+1}\dots dt_1dt\right\}$$

Tada imamo:

$$y(x,z) = \varphi(x,z) + \frac{1}{z}y(\beta_1,z)\psi(x,z)$$

Odakle, za $x = \beta_1$ imamo

$$y(\beta_1,z) = \frac{\varphi(\beta_1,z)}{1-\frac{1}{z}\psi(\beta_1,z)}$$

Dakle,

$$y(x,z) = \varphi(x,z) + \frac{\varphi(\beta_1,z)}{z-\xi(\beta_1,z)}\xi(x,z) \quad (5)$$

Then we have:

$$y(x,z) = \varphi(x,z) + \frac{1}{z}y(\beta_1,z)\psi(x,z)$$

where for $x = \beta_1$ we have

$$y(\beta_1,z) = \frac{\varphi(\beta_1,z)}{1-\frac{1}{z}\psi(\beta_1,z)}$$

Therefore,

$$y(x,z) = \varphi(x,z) + \frac{\varphi(\beta_1,z)}{z-\xi(\beta_1,z)}\xi(x,z) \quad (5)$$

Kako su $\varphi(0,z)=0$ i $\xi(0,z)=0$, to je uslov y (0) = 0 u funkciji (5) već ugrađen. Znači karakteristična jednačina našeg rubnog problema izgleda ovako

$$\varphi(\pi,z)+\frac{\varphi(\beta_1,z)}{z-\xi(\beta_1,z)}\xi(\pi,z)=0 \qquad (6)$$

Izučimo podrobno asimptotiku lijeve strane jednačine (6). Asimptotiku funkcije φ (x,y) data je u [7] , kao i asimptotika funkcije ζ (x,z) u slučaju konstantnog kašnjenja. I u slučaju ove jednačine postupajući analogno dobivamo:

Since $\varphi(0,z)=0$ and $\xi(0,z)=0$, it is a condition of y (0) = 0 in fuction (5) already installed. So the characteristic equation of the boundary problem looks like

$$\varphi(\pi,z)+\frac{\varphi(\beta_1,z)}{z-\xi(\beta_1,z)}\xi(\pi,z)=0 \qquad (6)$$

Thoroughly investigate the asymptotic left equation (6). Asymptotic function φ(x,z) is given in, and the asymptotic function ξ (x,z) in the case of constant delay. As in the case of our equation acting analogously we get:

$$\xi(x,z)=\sum_{i=0}^{\infty}\frac{\psi_{2i+1}(x)}{z^{2i+1}}+sinzx\sum_{\nu=1}^{\infty}\left[\frac{\psi_{2\nu}}{z^{2\nu}}\right]+\sum_{j=0}^{\infty}cosz\alpha_j(x)\sum_{\nu=1}^{\infty}\left[\frac{\psi_{j,\nu}^{(c)}(x)}{z^{\nu}}\right] \qquad (7)$$

gdje je $\alpha_j(x)=\alpha(\alpha\ldots\alpha(x)\ldots)$
$j=0,1,\ldots,\ \ \alpha_0(x)=x.$

Radi ilustracije navedimo nekoliko prvih koeficijenata:

$$\psi_1(x)=p_1(x)$$
$$\psi_{0,1}^{(c)}=-p_1(0)$$

$$\psi_3(x)=-p_1''^{(x)}+q(x)p(\alpha(x))$$
$$\psi_{0,2}^{(c)}=0$$

$$\psi_{0,3}^{(c)}=-p_1(0)q(0)\left[1+\frac{\alpha'(0)}{\alpha'^2(0)-1}\right],$$
$$\psi_{13}^{(c)}=\frac{p_1(0)q(0)}{\alpha'^2(x)-1},$$
$$\psi_2^{(3)}=-p_1(0)$$

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Uzet ćemo jednakosti iz (4) u nešto izmjenjenoj formi:

$$\xi(x,z)=\sum_{j=0}^{\infty}sin\alpha_j(x)z\left[\frac{1}{z^j}\sum_{\nu=0}^{\infty}\frac{A_{j\nu}(x)}{z^{\nu}}\right] \qquad (8)$$

$$A_{0,0}(x)=1 \qquad A_{1,0}(x)=0$$

$$A_{0,1}(x)=0 \qquad A_{1,1}(x)=0$$

where $\alpha_j(x)=\alpha(\alpha\ldots\alpha(x)\ldots)$
$j=0,1,\ldots,\ \ \alpha_0(x)=x.$

For the sake ilustration, said the first few coefficients

$$\psi_1(x)=p_1(x)$$
$$\psi_{0,1}^{(c)}=-p_1(0)$$

$$\psi_3(x)=-p_1''^{(x)}+q(x)p(\alpha(x))$$
$$\psi_{0,2}^{(c)}=0$$

$$\psi_{0,3}^{(c)}=-p_1(0)q(0)\left[1+\frac{\alpha'(0)}{\alpha'^2(0)-1}\right],$$
$$\psi_{13}^{(c)}=\frac{p_1(0)q(0)}{\alpha'^2(x)-1},$$
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We'll take the equity out of (4) in the form of something exchanged between a

$$\xi(x,z)=\sum_{j=0}^{\infty}sin\alpha_j(x)z\left[\frac{1}{z^j}\sum_{\nu=0}^{\infty}\frac{A_{j\nu}(x)}{z^{\nu}}\right] \qquad (8)$$

$$A_{0,0}(x)=1 \qquad A_{1,0}(x)=0$$

$$A_{0,1}(x)=0 \qquad A_{1,1}(x)=0$$

$$A_{0,2}(x) = -\frac{q(0)}{1-\alpha'^2(0)},$$
$$A_{0,2}(x) = \frac{q(x)}{1-\alpha'^2(x)}$$

Stavljajući $x = \pi$ i $x = \beta_i$ u (7) i (8) a potom uvrštavajući u (6), dobijam asimptotiku karakteristične funkcije. Karakteristična funkcija je meromorfna funkcija promjenjive z, gdje su nule funkcije z – ζ (β_1, z) polovi karakteristične funkcije. Slijedeći rezenovanje iz [7] karakterističnu funkciju F_{β_1}(z) možemo smatrati cijelom funkcijom. Takođe, rezonovanje iz [7] za slučaj $k > 1$ je i ovdje u potpunosti izvodljivo a i zaključci su analogni.

3. ZAKLJUČAK:

U navedenom radu posmatran je inverzni problem za Sturm-Liouville-ov operator sa otklonjenim argumentom, konstruisao sam njegovu karakterističnu funkciju, za diferencijalni operator zadat jednom sasvim novom jednačinom. Rezultati prethodno dokazane teoreme važe i za operator dat sa (9).

$$A_{0,2}(x) = -\frac{q(0)}{1-\alpha'^2(0)}, \quad A_{0,2}(x) = \frac{q(x)}{1-\alpha'^2(x)}$$

Puting $x = \pi i$ i $x = \beta_1$ in (7) and (8) and then by integrating the (6) we obtain the asymptotic characteristic function. Characteristic function is meromorphic function variable z, where z zero function z – ζ (β_1, z) poles characteristic function. Following the reasoning of [7] the characteristic function F_{β_1}((z) can be considered as the entire function. Also, the reasoning of [7] for the case k > l is here entirely feasible and conclusions are analog.

3. CONCLUSION

In this work I watched the inverse problem for the Sturm-Liouville's operator with outspread argument, I've built the characteristic function for the differential operator tenders, a completely new equation. Results previously proven theorems apply to the operator given by (9).

$$-y''(x) + \sum_{i=1}^{k} q_i(x) y(\alpha_i(x)) - \sum_{j=1}^{k} p_j(x) y(\beta_j) = \lambda y(x)$$
$$y(0) = y(x) = 0 \qquad (9)$$

Ovakva postavka i analiza predstavlja unaprjeđenje u poređenju sa modelima „idealnog" procesa koji se dobija pod pretpostavkom da poslije dejstva uopšte nema. Ovakva pretpostavka uglavnom dobro odražava realne pojave kao u slučajevima kada je kašnjenje vezano za predaju zvučnog signala, sa hidrauličnim udarom ili drugim talasnim procesima. Diferencijalni operatori sa konstantnim, homogenim kašnjenjem matematički modeliraju ove procese, a ovaj rad je jedan od tih matematičkih modela za navedene procese.

This item analysis and an improvement in comparison with the concept of "ideal" process which is obtained under the assumption that after the fact does not. This assumption generally well reflects the real world phenomena as in cases where the delay is related to the submission of a sound signal, with hydraulic shock wave or other processes. Differential operators with constant, homogeneous delay mathematical modeling of these processes, but this work is one of these mathematical models of these processes.

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